RECITATION 2

PRE-CALCULUS

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Section 1. *Distance*

§ 1 A. Absolute value

The absolute value of a real number x, $|x|$, is just x without the minus sign. But that's a syntactic notion, and isn't independent of notation. But intuitively, it's just x made positive. What this means is that if x is already positive, $|x|$ is just x, and if x is negative, we multiply by -1 to cancel the negative[i]: $|x|$ is $-x$. Of course, $x = 0$ is neither positive nor negative, but this isn't an issue, since intuitively $|0| = 0 = -0$. Formally, this definition is written as follows: for all real numbers x,

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

How we deal with absolute values can then be broken up into cases of whether the inside is positive or negative. The book likes to take short cuts, and that's nice if you remember them. But fundamentally, absolute value is a piecewise function, and we should deal with the cases separately. Let's start with a simple example.

1 A • 1. Example

$|x - 3| = 2x$ is equivalent to x being 1 or x being -3.

Proof .:.

We have two cases: either $x - 3$ is positive, or $x - 3$ is negative.

1. If $x - 3$ is positive, $|x - 3| = x - 3 = 2x$, which is equivalent to $x = -3$.
2. If $x - 3$ is negative, $|x - 3| = -(x - 3) = 2x$, which is equivalent to $x - 3 = -2x$ and thus $x = 1$. ⊣

More complicated equalities can be dealt with equivalently, just by breaking down into cases, and solving each separately. Inequalities can also be dealt with in this way.

1 A • 2. Example

$|x - 1| > 2$ is equivalent to x being greater than 3, or x being less than -1.

Proof .:.

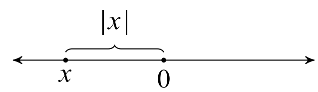
We have two cases: either $x - 1 \geq 0$, or $x - 1 < 0$.

1. If $x - 1 \geq 0$, then $|x - 1| = x - 1 > 2$ is equivalent to $x > 3$.
2. If $x - 1 < 0$, then $|x - 1| = -(x - 1) > 2$ is equivalent to $-x + 1 > 2$, which means $1 - 2 = -1 > x$.⊣

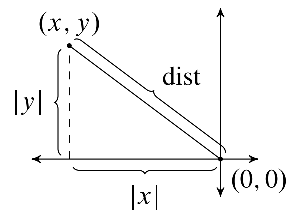
[i] Recall that $(-1)^2 = 1$.

§ 1 B. Distance

A slight generalization of absolute value is distance. For a real number x, $|x|$ can be seen as the distance between 0 and x on the real number line, as in the picture below.



When working in two dimensions, we can use the pythagorean theorem to determine the distance between the origin $(0, 0)$ and a point (x, y), as in the picture below: $\text{dist}^2 = |x|^2 + |y|^2 = x^2 + y^2$ implies that, dist is either $\sqrt{x^2 + y^2}$ or $-\sqrt{x^2 + y^2}$. Since dist is positive, we get that dist $= \sqrt{x^2 + y^2}$.



If we want to find the distance between a point (x_0, y_0) and a point (x_1, y_1) other than the origin, all we have to do is shift the axes: consider the points $(x_0 - x_1, y_0 - y_1)$ and $(x_1 - x_1, y_1 - y_1) = (0, 0)$. Distance is preserved under such translations, so we can calculate the distance just as before.

Distance is especially useful in talking about circles, which are just defined as the collection of points which are at a fixed distance (the radius) from some fixed point (the center). There are all sorts of trigonometric questions one can ask about this. For now, just remember the general equation for a circle of radius r, and center $(0, 0)$: $x^2 + y^2 = r^2$. This just comes from the fact that (x, y) being on the circle requires the distance from $(0, 0)$ to be r, i.e. $\sqrt{x^2 + y^2} = r$, meaning $x^2 + y^2 = r^2$. If we want a different center, (x_0, y_0), we again just shift the axes: $(x - x_0)^2 + (y - y_0)^2 = r^2$.

Section 2. Lines

There are two things which uniquely describe a line: a point, and a direction. Without specifying a point, you can get many lines going in the same direction, like stripes. Without specifying a direction, you can get many lines going through the same point, like a cross. But once you specify a point, and a direction, there's no wiggle room: the line is determined.

2 • 1. Definition

The direction of a line in terms of the the change of x is the *slope*, defined by, for points on the line, slope $= \dfrac{\text{the change in } y}{\text{a change in } x}$. The notation Δ is sometimes used to denote change or difference: slope $= \Delta y / \Delta x$.

As a result, if we have two points, that gives us our direction, and a point, and thus two points determine a line. Note that the slope may not be defined for any given line, since there might be no change in x between points on the line. For example, $x = 2$ describes a vertical line[ii], but for any (x_0, y_0) and (x_1, y_1) on the line, the change in x is $x_0 - x_1 = 2 - 2 = 0$.

[ii] y can be anything, and the equation is true. So the set of points (x, y) that make the equation true consists of points $(2, y)$ for any y.

2 • 2. Definition

When the slope is defined, we can write the equation $y - y_0 = \text{slope} \cdot (x - x_0)$ for any point (x_0, y_0) on the line. This is called *point-slope form*.
This also yields *slope-intercept form*: $y = \text{slope} \cdot x + b$ for some b, the y-intercept (where the line meets the y-axis, i.e. the value of y when $x = 0$).

A more neutral way to write a line without reference to slope (so as to include vertical lines) is the standard form: $(\Delta y) \cdot y + (\Delta x) \cdot x + m = 0$ for some, fixed number m.

§ 2 A. Parametric definitions of lines

Sometimes both x and y can be seen as functions of time, t, in that $x(t) = \text{slope}_x \cdot t + x_0$, and $y(t) = \text{slope}_y \cdot t + y_0$. If we need a description of y in terms of x rather than t, we can easily solve for t in terms of xy, and then plug this back into the equation of y to get a description of y in terms of x. The issue with this, beyond being laborious, is that it takes a while. More directly, we have the following result.

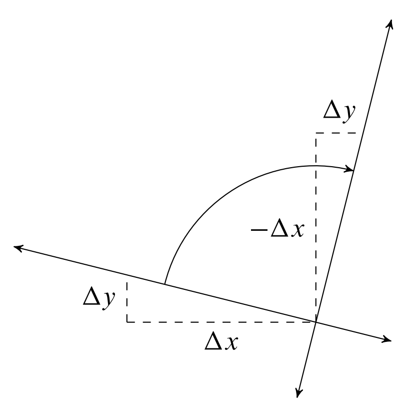
2 A • 1. Result

Let $x(t) = \text{slope}_x \cdot t + x_0$, and $y(t) = \text{slope}_y \cdot t + y_0$. Suppose $\text{slope}_x \neq 0$.
Therefore, for the line in terms of x and y, if $\text{slope} = \frac{\text{slope}_y}{\text{slope}_x}$, and (x_0, y_0) is a point on the line. Hence

$$y - y_0 = \frac{\text{slope}_y}{\text{slope}_x}(x - x_0).$$

§ 2 B. Parallel and perpendicular lines

A line parallel to another is just one with the same slope, but different points. The line perpendicular to another has been rotated: the xs and ys have been switched, and reversed, as in the picture below: the new change in x is the old Δy, and the new change in y is the old $-\Delta x$.



2 B • 1. Result

Let $y = \frac{\Delta y}{\Delta x}x + b$ be a line. Therefore any parallel line has slope $\frac{\Delta y}{\Delta x}$, and any perpendicular line has slope $\frac{-\Delta x}{\Delta y}$.

Hence for any line, and any point (not necessarily on the line), we can find a perpendicular line going through that point.

Section 3. *Exponentials and Logarithms*

Recall from basic arithmetic that exponents are defined as repeated multiplication. While this makes computation relatively easy, such a view cannot be held with respect to all real numbers. The expansion of exponents to the rational numbers (i.e. fractions or *ratio*s) already involves the addition of the concept of taking roots: $4^{1/2} = 2$, for example. The idea is that $x^{a/b}$ is just the number y that satisfies $y^b = x^a$. Since there may be multiple such y, we choose the non-negative one. This expansion still is not enough to capture how to compute, say, 2^π. The precise definition of this deals with a so-called *limit* of approximations by rational exponents in the same way that π is the limit of the sequence 3, 3.1, 3.14, 3.141, $\cdots$; 2^π is the limit of the sequence 2^3, $2^{3.1}$, and so on. So direct calculation is nearly impossible for exponents that aren't nice and neat numbers.

3 • 1. Definition

For n a natural number, and x a real number, $x^n = x \cdot \ldots \cdot x$, being x multiplied n times. $x^{-n} = 1/x^n$.
For a/b a rational number, and x a real number, $x^{a/b} = y$ where y is the positive number (or 0) satisfying $x^a = y^b$.
For x and r real numbers, x^r is the limit of the approximations x^{r_n} where each r_n is the first n digits of r.

We can immediately check certain properties of this. Note that (4) follows from (1) and (2) since $r - s = r + (-s)$.

3 • 2. Result

Let x, r, and s be real numbers. Therefore,

1. $x^{r+s} = x^r \cdot x^s$;
2. $x^{-r} = 1/x^r$;
3. $(x^r)^s = x^{r \cdot s}$; and
4. $x^{r-s} = x^r/x^s$.

Now we introduce the logarithm, which is just the inverse of the exponential: $\log_a(a^x) = x$. So for each a, we get a separate logarithm. Generally, log on its own stands for either $\log_{10}$ or $\log_e = \ln$, depending on the situation. In mathematics, $\log = \ln$. In less pure areas, $\log = \log_{10}$. In particular, log is defined in terms of the exponential as an inverse.

3 • 3. Definition

Let x and r be real numbers. Define $\log_r(x)$ as the y such that $x = r^y$. In essence, $\log_r(x)$ is supposed to be the exponent of r, and in particular, it's the one with $x = r^{\log_r(x)}$.

As a result of this definition, we immediately get some results which follow from Result 3 • 2

3 • 4. Result

Let x, y, r, and s be real numbers. Therefore,

1. $\log_r(x \cdot y) = \log_r(x) + \log_r(y)$;
2. $\log_r(x/y) = \log_r(x) - \log_r(y)$;
3. $\log_r(x^y) = y \cdot \log_r(x)$;
4. $\frac{\log_r(x)}{\log_r(s)} = \log_s(x)$.

Proof .:.

1. $\log_r(x \cdot y)$ has $r^{\log_r(x \cdot y)} = x \cdot y = r^{\log_r(x)} \cdot r^{\log_r(y)} = r^{\log_r(x)+\log_r(y)}$. Hence the result holds.
2. This follows from (1) and (3), since $x/y = x \cdot y^{-1}$:
$$\log_r(x/y) = \log_r(x) + \log_r(y^{-1}) = \log_r(x) - \log_r(y).$$

3. $\log_r(x^y)$ has $r^{\log_r(x^y)} = x^y = (r^{\log_r(x)})^y = r^{y\log_r(x)}$. Hence the result holds.
4. $\log_r(x)$ has
$$r^{\log_r(x)} = x = s^{\log_s(x)} = (r^{\log_r(s)})^{\log_s(x)} = r^{\log_s(x)\cdot\log_r(s)}.$$
Hence $\log_r(x) = \log_s(x)\cdot\log_r(s)$ and thus $\frac{\log_r(x)}{\log_r(s)} = \log_s(x)$. ⊣

Section 4. *Exercises*

Exercise 1

Find the xs for which $|x| < -1$.

Solution .:.

This never happens, since $|x|$ is always at least 0. We can also show this formally: $|x| < -1$ implies that either $x \geq 0$ and $x < -1$ (impossible), or $x \leq 0$ and $|x| = -x < -1$, which is equivalent to $x \leq 0$ while $1 < x$ (again, impossible). So there is no possible way for this to be true.

Exercise 2

Define the function f by
$$f(x) = \begin{cases} x+2 & \text{if } x > 3 \\ |x| & \text{if } x \leq 3. \end{cases}$$
Determine which x have $f(x) = 8$. Determine which x have $f(x) = -1$.

Solution .:.

If $x > 3$, then $f(x) = x + 2 = 8$ if and only if $x = 6$. If $x \leq 3$, then $f(x) = 8$ iff $|x| = 8$, which happens if and only if $x \leq 3$ and either $x = 8$ or $x = -8$. Since $x \leq 3$ in this case, we can't have $x = 8$, and so $x = -8$ is the only possibility. Hence $x = 6$ and $x = -8$ are the only solutions.

If $x > 3$, then $x + 2 = -1$ requires $x = -3 \not> 3$. So no such x can occur after 3. If $x \leq 3$, then $f(x) = |x| \geq 0 > -1$. So again, in this case, x cannot exist. Therefore there is no x for which $f(x) = -1$.

Exercise 3

Evaluate $\log_4(2) + \log_2(4)$.

Solution .:.

Firstly, $\log_4(2) = \log_4(4^{1/2}) = \frac{1}{2}\log_4(4) = \frac{1}{2}$. Secondly, $\log_2(4) = \log_2(2^2) = 2$ by the same reasoning. Therefore the sum is $5/2$.

Exercise 4

Graph e^x. Graph e^{x-5}. Graph e^{x+5}. Graph e^{2x}. Graph $e^{x/2}$.

Exercise 5

In general, for a function f, how do these compositions change the graph of f? In other words, how does the graph of g compare to the graph of f?

1. $g(x) = f(x + 10)$;
2. $g(x) = f(x) + 10$;
3. $g(x) = f(3x)$;
4. $g(x) = 3f(x)$;
5. $g(x) = f(3x + 5)$.

Solution .:.

1. This shifts the graph of f to the *left* by 10, since at any x, we consider the value of f 10 to the right of x, and place it back on the left with x.
2. This shifts the graph of f up by 10, since we evaluate f normally, and then add 10 to the vertical part.
3. This squeezes the graph of f horizontally, since a small change in x registers as a bigger change in $3x$, which is where we evaluate f.
4. This stretches the graph of f vertically, since a small change in f registers as a bigger change in g.
5. This shifts the graph of f by $5/3$ units to the left, and squeezes the graph of f horizontally, reducing it by a third. This is because $3x + 5 = 3(x + 5/3)$. The 3 out in front squeezes the graph of f, and $x + 5/3$ shifts the graph of $f(3x)$ by $5/3$s to the left by (1).

Exercise 6

Find the line passing through $(2, 3)$ that is perpendicular to the line passing through $(-1, 2)$, $(4, 2)$.

Solution .:.

The line through $(-1, 2)$ and $(4, 2)$ has a slope of $\frac{2-2}{-1-4} = 0$. Hence the line is horizonal: $y = 0 \cdot x + 2$. The line perpendicular to this has no slope: $\frac{1+4}{0}$ doesn't exist. So the perpendicular line is vertical, and thus is of the form $x = 2$, since it passes through $(2, \text{something})$.

Exercise 7

Evaluate $e^{(\log_{e^3}(8))}$.

Proof .:.

Note that $e = (e^3)^{1/3}$, so that

$$\begin{aligned} e^{(\log_{e^3}(8))} &= \left((e^3)^{\frac{1}{3}}\right)^{\log_{e^3}(8)} \\ &= (e^3)^{\frac{1}{3}\log_{e^3}(8)} && \text{by (3) of Result 3 • 2} \\ &= \left((e^3)^{\log_{e^3}(8)}\right)^{\frac{1}{3}} && \text{by (3) of Result 3 • 2} \\ &= (8)^{\frac{1}{3}} && \text{by Definition 3 • 3} \\ &= 2. \end{aligned}$$